Space-Charge Wave Harmonics and Noise Propagation in Rotating Electron Beams

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Higher-order space-charge waves on solid cylindrical electron beams produced by shielded or nearly shielded guns have only azimuthal periodicity, as in hollow beams. Because of beam rotation, they are members of a broad class of space-charge waves which can travel faster than the beams themselves, either forwards or backwards. The properties of such waves for the beam in a drift tube and in a concentric sheath helix are derived from a slow-wave, small-signal analysis and the appropriate boundary equations. Experimental observations of their interaction with harmonic fields of a helix, as well as of their role in noise propagation, tend to confirm the results of these computations.

I. INTRODUCTION

Interest in the ac behavior of cylindrical electron beams issuing from magnetically shielded or partly shielded guns has been stimulated in recent years by their increasing application in medium- and high-power traveling-wave tubes. As yet, however, such beams have received considerably less attention in the literature than have those in confined flow. The properties of the fundamental (axial-symmetric) space-charge mode in the former type of beam have been studied by Rigrod and Lewis,[1] and by Brewer.[2] Waves of this type provide a first-order description of the beam interaction with its environment, such as a drift tube or helix. The present paper will supplement this work by considering higher-order modes of wave propagation in such beams, in which the fields have azimuthal, but not radial, periodicity. Following an analysis of the waves themselves, several problems will be discussed in which they play important roles: the excitation in a helix of spatial-harmonic modes, the propagation of noise excitation and possible new

applications of these space-charge-wave "harmonics." Experimental confirmation of their interaction with the harmonic fields of a helix, observed by Kiryushin[3,4] will be described, as well as some interesting noise measurements obtained by Ashkin and White,[5] which illustrate their participation in noise propagation.

II. NATURE OF HIGHER-ORDER MODES

The formation of ripple-free beams from convergent electron guns is often facilitated by letting some magnetic flux thread the cathode. Although this paper is primarily concerned with waves along Brillouin-flow beams, the computations of this section will include provision for arbitrary flux density at the cathode, for greater generality.* The ratio α of flux encircled at the cathode to that in the drift region, is assumed constant for any ring of electrons. The steady-state electron flow is then laminar, and can be described by the following equations:

$$\dot{\theta} = \frac{\omega_c}{2}(1 - \alpha), \tag{1}$$

$$\eta \frac{\partial V_0}{\partial r} = r\dot{\theta}(\omega_c - \dot{\theta}), \tag{2}$$

$$\omega_p^2 = \omega_c^2 \frac{(1 - \alpha^2)}{2}, \tag{3}$$

$$\dot{z} = u, \quad \dot{r} = 0. \tag{4}$$

Here (r, θ, z) are polar cylindrical coordinates; V_0 the dc potential due to the uniform space-charge density ρ_0 ; η the charge-mass ratio for the electron (a positive quantity) and ω_c and ω_p the angular cyclotron and plasma frequencies, respectively. A dot indicates time differentiation, and MKS units are used.

The problem is to find the properties of small-signal ac waves which propagate along the beam as

$$\exp j(\omega t - n\theta - \beta z), \tag{5}$$

with $n = 0, 1, 2, \cdots$, subject to the slow-wave condition

$$\frac{\omega^2 \mu\epsilon}{\beta^2} = \frac{k^2}{\beta^2} \cong 0. \tag{6}$$

With this condition, the scalar wave equation

* The basic equations of this section were first derived by J. R. Pierce of Bell Telephone Laboratories.

$$(\Delta + k^2)E_z = \frac{\beta}{\omega\epsilon}\,\mathrm{div}\,\mathbf{J} + j\omega\mu\,J_z \tag{7}$$

reduces to the following equivalent forms

$$\Delta E_z = \frac{\beta}{\omega\epsilon}\,\mathrm{div}\,\mathbf{J}, \tag{8}$$

$$\Delta E_z = -\frac{j\beta\rho}{\epsilon}, \tag{9}$$

using the charge-conservation equation. Here Δ is the Laplacian operator, $\mathbf{J}$ the ac convection current density, ρ the ac space-charge density, ϵ and μ the dielectric constant and permeability of free space, respectively, ω the angular excitation frequency, k the free-space wave number and β the axial propagation constant.

The terms which drop out of this wave equation due to the slow-wave assumption are precisely those arising from curl $\mathbf{E}$. That is, the slow-wave condition is equivalent to setting curl $\mathbf{E}$ to zero, or to neglecting the contribution to $\mathbf{E}$ made by the ac magnetic fields (provided $\mathbf{J}$ does not exceed $j\omega\epsilon\mathbf{E}$ by a factor approaching β^2/k^2 in magnitude). The electric field can therefore be derived from a scalar potential, or

$$E_r = \frac{j}{\beta}\frac{\partial E_z}{\partial r}, \qquad E_\theta = \frac{n}{\beta r}E_z. \tag{10}$$

Another consequence of the slow-wave restriction is that the contribution of the ac magnetic field to the force on electrons can be disregarded, as it is negligible compared with that exerted by the electric field.

With this and the assumption of single-valued velocities at each point in the beam, the electron dynamics equation can be expressed in Eulerian coordinates as follows:

$$\frac{d}{dt}(\mathbf{v}_0 + \mathbf{v}) = -\eta[-\mathrm{grad}\,V_0 + \mathbf{E} + (\mathbf{v}_0 + \mathbf{v}) \times \mathbf{B}_0], \tag{11}$$

where

$$\mathbf{v}_0 = (0, r\dot{\theta}, u), \tag{12}$$

$$\mathbf{v} = (v_r, v_\theta, v_z)\exp j(\omega t - n\theta - \beta z), \tag{13}$$

and $\mathbf{B}_0$ is the axial magnetic field, the zero subscript being used wherever necessary to distinguish the steady-state quantities. Expansion of this equation yields the components of the ac velocity amplitude:

$$v_r = \frac{j\eta P}{\omega_n^2}\left[E_r + j\left(\frac{\alpha\omega_c}{P}\right)E_\theta\right], \tag{14}$$

$$v_\theta = \frac{j\eta P}{\omega_n^2}\left[E_\theta - j\left(\frac{\alpha\omega_c}{P}\right)E_r\right], \tag{15}$$

$$v_z = \frac{j\eta E_z}{P}, \tag{16}$$

where

$$P = \omega - n\dot{\theta} - \beta u, \tag{17}$$

and

$$\omega_n^2 = P^2 - (\alpha\omega_c)^2. \tag{18}$$

From the charge-conservation equation,

$$\rho = \frac{j\rho_0 \operatorname{div} \mathbf{v}}{P} = \frac{j\epsilon\omega_p^2}{\omega_n^2}\left[\Delta E_z + \left(\frac{\alpha\omega_c}{P}\right)^2 \beta^2 E_z\right], \tag{19}$$

and the wave equation for E_z reduces to the Bessel equation

$$\left(1 - \frac{\omega_p^2}{\omega_n^2}\right)\Delta E_z - \frac{\omega_p^2}{\omega_n^2}\left(\frac{\alpha\omega_c}{P}\right)^2 \beta^2 E_z = 0, \tag{20}$$

whose solution has the form

$$E_z = \sum_n A_n I_n(\gamma_n r) \exp j(\omega t - n\theta - \beta_n z). \tag{21}$$

Here A_n is a constant, and I_n the nth order modified Bessel function of the first kind, with transverse propagation constant γ_n defined by

$$\frac{\gamma_n^2}{\beta_n^2} = 1 + \left(\frac{\alpha\omega_c}{P}\right)^2\left(\frac{\omega_p^2}{\omega_n^2 - \omega_p^2}\right). \tag{22}$$

The ac space-charge density can conveniently be re-expressed in terms of the above ratio (for any chosen n):

$$\rho = \frac{j\epsilon\Delta E_z}{\beta} = j\epsilon\beta\left(\frac{\gamma^2}{\beta^2} - 1\right)E_z, \tag{23}$$

showing that ρ becomes zero when $\alpha\omega_c$ is zero.

Since, for slow waves, the electric field is irrotational both inside and outside of the beam, it can be determined by the boundary conditions for E_z and E_r at the beam surface:

$$\left(\frac{E_r + \tilde{\sigma}/\epsilon}{E_z}\right)_{r=b-} = \left(\frac{E_r}{E_z}\right)_{r=b+}, \tag{24}$$

where $b-$ and $b+$ refer to the regions just inside and outside of the beam surface, respectively, and $\tilde{\sigma}$ is the ac surface-charge density due to unbalanced radial electron motions:

$$\tilde{\sigma} = -\left(\frac{j\rho_0 v_r}{P}\right)_{r=b-} = -\frac{j\epsilon\omega_p^2}{\beta\omega_n^2}\left[\frac{\partial E_z}{\partial r} + \left(\frac{\alpha\omega_c}{P}\right)\frac{n}{r}E_z\right]_{r=b-}. \qquad (25)$$

The simplest boundary-value problem is that of the beam drifting in a concentric conducting tube of radius a. In the space between beam and tube wall, the field is of the form

$$E_z = B_n[I_n(\beta r)K_n(\beta a) - K_n(\beta r)I_n(\beta a)], \qquad (26)$$

where B_n is a constant, and K_n the nth order modified Bessel function of the second kind. Thus, the boundary equation at the beam surface, $r = b$, can be written

$$\left[\left(1 - \frac{\omega_p^2}{\omega_n^2}\right)\frac{\gamma b I_n'(\gamma b)}{I_n(\gamma b)} - \frac{\omega_p^2}{\omega_n^2}\frac{n\alpha\omega_c}{P}\right] = \beta b\left[\frac{I_n'(\beta b)K_n(\beta a) - K_n'(\beta b)I_n(\beta a)}{I_n(\beta b)K_n(\beta a) - K_n(\beta b)I_n(\beta a)}\right], \qquad (27)$$

the primes denoting differentiation with respect to the total argument. For any set of values of n, α and b/a, this equation can be solved for the square of the plasma-frequency reduction factor $p_n = P/\omega_p$. For each frequency, there are two values of the propagation constant:

$$\beta_{1,2} = \beta_e - n\dot{\theta}/u \pm p_n\beta_p, \qquad (28)$$

where $\beta_e = \omega/u$, $\beta_p = \omega_p/u$, and p_n is a function of βb. The two traveling waves in each such solution interfere with one another to form a standing wave, with half-wavelength

$$\frac{\lambda_s}{2} = \frac{2\pi}{\beta_1 - \beta_2} \cong \frac{\pi}{p_n\beta_p}. \qquad (29)$$

Brewer[2] has solved this admittance equation (27) for the fundamental mode, $n = 0$, using a flux parameter Ω related to α by

$$\left(\frac{\Omega}{\omega_p}\right)^2 = \frac{\alpha^2}{2(1 - \alpha^2)}. \qquad (30)$$

His results show that, for α below about 0.5, the solution p_0 differs little from its value for $\alpha = 0$, the rate of change $dp_0/d\alpha$ being less as βb and b/a decrease.

The influence of cathode flux on the reduction factor p_n for the higher-order modes is quite different from that for the fundamental. This is

illustrated in Fig. 1, showing how this factor varies with βb for the $n = 1$ mode, for $\alpha = 0.2$ and 0.4, and $b/a = 0$ and 0.6. For $0 < \alpha < 1$, and small b/a, p tends to increase as βb decreases, reaching some finite value at the limit $\beta b = 0$. Calculations indicate, moreover, that p becomes infinite for $\alpha = 1$ (confined flow), for all values of βb. (In confined flow, there is an infinite set of solutions for p, but the one described here is that which blends continuously into that for Brillouin flow as α is varied from unity to zero.) In general, $dp/d\alpha$ decreases as βb increases or α decreases.

In most cases when beams are produced by shielded or nearly shielded diodes, the flux parameter α ranges from zero to at most about 0.4. Except for very small βb and b/a, the reduction factor for $\alpha = 0.2$ differs negligibly from that for $\alpha = 0$, and for $\alpha = 0.4$ it ranges mostly between 0.85 and unity. Over this range of α, then, it would appear that the

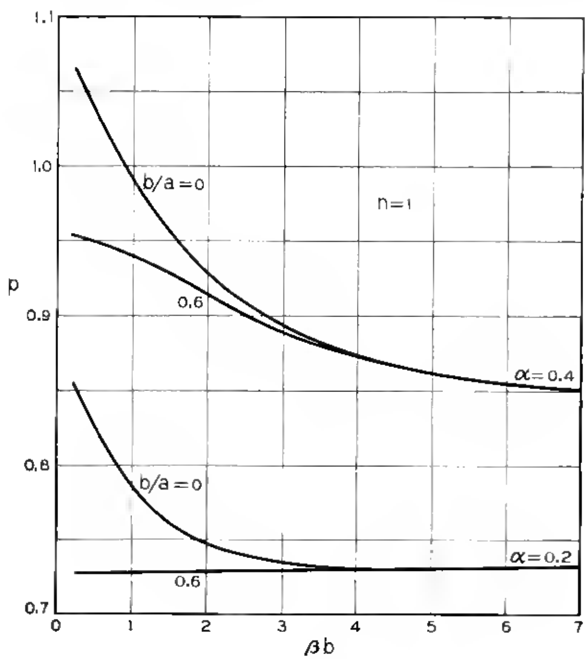

Fig. 1 — Plasma-frequency reduction factors $p = \omega_q/\omega_p$ for space-charge waves with azimuthal periodicity, $n = 1$, along a solid-cylindrical beam with small amounts of flux threading the cathode; α is the ratio of flux at the cathode to that flooding the beam, β is the axial propagation constant, b the beam radius and a the radius of a concentric drift tube.

properties of higher-order space-charge waves do not differ markedly from those on a Brillouin-flow beam (Figs. 2 and 3).

To obtain the equations for Brillouin flow, it is only necessary to set α equal to zero. It should be noted that, in all of the functional relations among ac quantities, the flux parameter α appears explicitly only in the *product* $\alpha\omega_c$, proportional to the flux density at the cathode. Because of this, all the equations determining the current and field patterns of the higher-order modes, including the TM boundary-matching equation, are the same for the Brillouin-flow beam and the beam in zero magnetic field, both of which have zero flux at the cathode. The only wave properties affected by the rotation of the Brillouin-flow beam are the θ-directed surface current (which excites TE fields), and the axial phase velocity of higher-order modes.

The reduced space-charge wavelength, however, is the same in both types of beams. In a shielded diode with small convergence angle, therefore, the accelerated beam throughout the univelocity region can be regarded approximately as a chain of short sections of drifting beams, each with its own velocity and geometry, in which the allowed mode patterns are the same as in Brillouin flow. When the beam enters the magnetic focusing field, these patterns rotate with the rotating beam, each thereby acquiring a higher axial phase velocity.

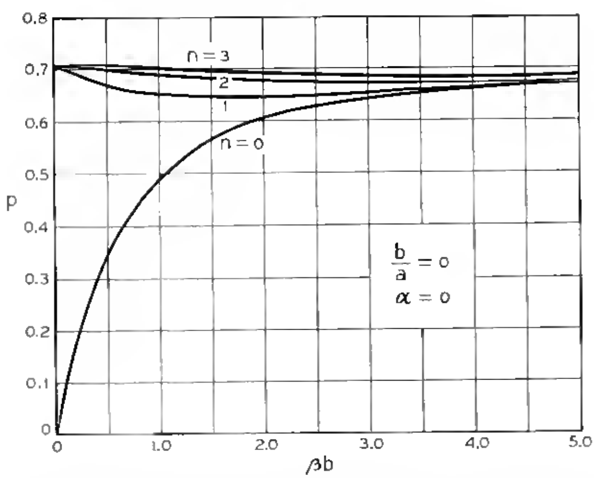

Fig. 2. — Plasma-frequency reduction factors $p = \omega_q/\omega_p$ for the fundamental ($n = 0$) and the first three higher-order modes of space-charge waves, along a solid-cylindrical Brillouin-flow beam ($\alpha = 0$), of radius b in free space.

When $\alpha\omega_c$ is zero, the wave equation for E_z has two sets of solutions. The first is

$$\omega_p^2 = P^2 \quad \text{or} \quad p_n = 1. \tag{31}$$

It has been shown in the accompanying article[6] that this solution is spurious, as there is no way in which the corresponding waves can be excited. The second solution is

$$\Delta E_z = 0. \tag{32}$$

The transverse propagation constant is now $\gamma = \beta$, and the ac space-charge density in the beam is zero. The boundary equation can be reduced to an explicit expression for the space-charge reduction factor:

$$p_n^2 = \beta b I_{nb}' I_{nb} \left[\frac{K_{nb}}{I_{nb}} - \frac{K_{na}}{I_{na}} \right]. \tag{33}$$

Here, and wherever else they are unambiguous, the arguments (βa) and (βb) are replaced by the subscripts a and b, the radii of drift tube and beam, respectively.

For very small arguments, $\beta a < n^2$, and $n > 0$,

$$p_n^2 \cong \tfrac{1}{2} - \tfrac{1}{2}(b/a)^{2n}, \tag{34}$$

whereas, for very large arguments, $\beta b > n^2$, and $n \geqq 0$,

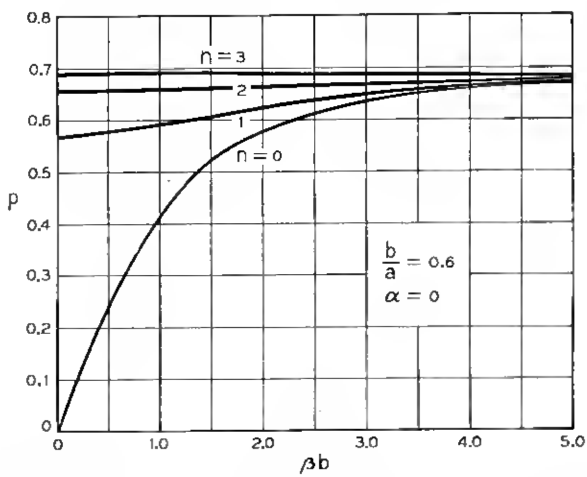

Fig. 3. — Plasma-frequency reduction factors $p = \omega_q/\omega_p$ for the same modes as in Fig. 2, when the Brillouin-flow beam is in a concentric drift tube of radius a, where $b/a = 0.6$.

$$p_n^2 \cong \tfrac{1}{2} - \tfrac{1}{2} \exp [-2\beta(a - b)]. \tag{35}$$

These limiting expressions, and the curves of p_n plotted in Figs. 2 and 3 for $n = 1$, 2 and 3, for two values of b/a, show that the smaller b/a is and the larger n is the closer the reduction factor p_n clings to the asymptotic value 0.707. To a good approximation, then, for all $n > 1$, and for $n = 1$ when b/a is small,

$$\beta_n \cong \beta_e - \tfrac{1}{2}(n \mp 1)\beta_c \tag{36}$$

and

$$\frac{\lambda_s}{2} \cong \frac{2\pi}{\beta_c} = \lambda_c ; \tag{37}$$

that is, the distance between current minima equals the cyclotron wavelength.

Since the convection current carried by the beam is chiefly due to transport of ac surface charge, the field pattern can best be visualized by examining the locus of maximum surface current density in each standing wave:

$$\mathrm{Re}(G_s + G_f) = | G | \cos (\omega t - \beta_e z) \cos n \left(\theta - \frac{\dot{\theta} z}{u}\right) \cos \beta_q z. \tag{38}$$

The subscripts s and f refer to the slow and fast waves, respectively, and $| G |$ is the amplitude of surface current density at $t = z = \theta = 0$. For most of the high-order modes, for which $p_n \cong 0.707$,

$$\beta_q z \cong \frac{1}{2} \beta_c z = \frac{\dot{\theta} z}{u} . \tag{39}$$

The surface-current maxima follow spiral loci, therefore, with the same "pitch" as the rotating beam itself, increasing and decreasing along these loci with a period equal to one beam rotation. For the nth order mode, there are $2n$ such loci, resembling the conductors of a multifilar helix, in which the lines of force start and end on ac charges in adjacent parts of the beam. This is why the reduction factor tends to be independent of beam or wall geometry when n is large.

III. MODE COUPLING BETWEEN BEAM AND HELIX

The coupling impedance, measuring the interaction between waves on a Brillouin-flow beam and a concentric sheath helix, both with the same azimuthal periodicity, will be evaluated in this section, with some simplifying assumptions. Somewhat weaker interaction should also be

possible when the beam and helix waves have the same axial phase velocities, but different azimuthal periodicities. However, this problem will not be treated here.

The four boundary-matching equations at a sheath helix[1,7] involve E_z, E_θ, H_z and H_θ, as well as the helix pitch angle Ψ. In addition to matching E_z and its radial derivative at the beam boundary, therefore, it would seem necessary to introduce another two equations in terms of H_z and its radial derivative at the beam boundary:

$$(H_z + G_\theta)_{b-} = (H_z)_{b+}, \tag{40}$$

$$\left(\frac{\partial H_z}{\partial r} + J_\theta\right)_{b-} = \left(\frac{\partial H_z}{\partial r}\right)_{b+}. \tag{41}$$

From the results of the previous section, it is readily found that

$$\Delta H_z = -\operatorname{curl}_z \mathbf{J} = 0. \tag{42}$$

Thus the radial propagation constant for H_z is β, inside as well as outside of the beam.

The eliminant of the eight boundary-matching equations can be combined in the form of a single wave-admittance equation, similar in form to that for the beam in a drift tube:

$$\left(1 - \frac{\omega_p^2}{P^2}\right)\frac{I_{nb}'}{I_{nb}} = \frac{I_{nb}' + \delta K_{nb}'}{I_{nb} + \delta K_{nb}}, \tag{43}$$

where δ stands for an expression which depends on the helix geometry and the amplitude of H_z. When the slow-wave assumption is invoked, however, it turns out that this term differs negligibly from its value when the beam is absent; i.e., the TE-TM wave coupling at the helix is negligibly small:

$$\delta \cong \delta_0 = -\frac{1}{K_{na}^2}\left[I_{na}K_{na} + \left(\frac{ka \cot \Psi}{\beta a + n \cot \Psi}\right)^2 I_{na}'K_{na}'\right]. \tag{44}$$

The TE fields excited by J_θ and G_θ in the beam, therefore, do not affect the TM wave admittance presented to the beam by the helix; the expression on the right-hand side of (43) would be the same whether or not the beam rotated. In addition, due to the absence of ac space charge inside of the beam, the field there has the same radial variation as it would have in free space. These two circumstances suggest the possibility of evaluating the normal-mode parameters[7] of the Brillouin-flow beam in terms of an "equivalent" thin hollow beam in confined flow, by the same method employed earlier for the axial-symmetric mode.[1]

The procedure consists of reformulating (43) by replacing the electronic admittance of the Brillouin beam with that of a thin hollow confined-flow beam of the same radius b, direct current I_0 and longitudinal velocity u. (It is not convenient to compare these beams on the basis of the same dc voltage.) The altered circuit admittance Y_B, consistent with this new electronic admittance and (43), is compared with the actual circuit admittance of a thin hollow beam, in a narrow range of propagation constants near that of the empty helix. The normal-mode parameters of each beam, which can then be compared, indicate how the different distributions of electron flow in the two beams affect their interaction with the helix field near synchronism. (The compared beams are "equivalent" only in their common dc properties, not their RF behavior.)

The beam admittance on the left-hand side of (43) has two components, one due to the displacement current and another due to the electron current, within radius b. The latter portion, the electronic beam admittance, is Y_e in the following restatement of (43):

$$Y_e = Y_c \tag{45}$$

$$-\frac{\omega_p^2}{(\omega - \beta u - n\dot{\theta})^2}\frac{I_{nb}'}{I_{nb}} = \frac{I_{nb}' + \delta_0 K_{nb}'}{I_{nb} + \delta_0 K_{nb}} - \frac{I_{nb}'}{I_{nb}}. \tag{46}$$

The expression on the right side, Y_c, is the net circuit admittance due to displacement current both inside and outside of the Brillouin-flow beam.

The boundary equation for a thin hollow beam (thickness dr) at its outer radius b is obtained by matching the free-space values of E_z inside and outside of the beam, and equating the change in H_θ, between these surfaces, to $J_z dr$. (Inside of this beam, the field E_z is taken to be the same as in free space.) With $b-$ to identify the fields just inside of this beam, and $b+$ the fields outside of it, this boundary equation is

$$\frac{J_z\, dr}{E_z} = \left(\frac{H_\theta}{E_z}\right)_{b+} - \left(\frac{H_\theta}{E_z}\right)_{b-}. \tag{47}$$

For confined flow and fields with azimuthal periodicity n, this reduces to

$$-\frac{\omega_{pH}^2 \beta dr}{(\omega - \beta u)^2} = \frac{I_{nb}' + \delta_0 K_{nb}'}{I_{nb} + \delta_0 K_{nb}} - \frac{I_{nb}'}{I_{nb}} = Y_c, \tag{48}$$

where ω_{pH} is the angular plasma frequency in the hollow beam. The expression on the left-hand side is the electronic admittance of the thin hollow beam, and that on the right is the net circuit admittance due to the helix, the same as in (46) for the solid Brillouin beam.

If the hollow beam is assigned the same direct current and longitudinal velocity as the Brillouin-flow beam, the plasma frequencies in the two beams are related by

$$\frac{{\omega_{pH}}^2}{{\omega_{pB}}^2} = \frac{\pi b^2}{2\pi b\, dr} = \frac{b}{2\, dr}. \tag{49}$$

Then the admittance equation (46) for the Brillouin-flow beam can be rewritten as though it were a thin hollow beam whose circuit admittance was

$$Y_B = \left[\frac{\beta b}{2}\left(\frac{\omega - \beta u - n\dot{\theta}}{\omega - \beta u}\right)^2 \frac{I_{nb}}{I_{nb}'}\right] Y_c\,. \tag{50}$$

The solid-cylindrical Brillouin beam is thus equivalent to a confined-flow hollow beam whose helix admittance is Y_B. The normal-mode parameters of this and a true hollow beam depend on the behavior of Y_B and Y_c in the neighborhood of the synchronous phase velocity, i.e., of their common zero and pole. As the latter are very close together, the admittance functions can be represented in this region by a Weierstrass (algebraic) approximation. Then, just as in the case of the axial-symmetric mode,[1,8] the two types of beam have the same space-charge parameter:

$$Q_B = Q_H\,, \tag{51}$$

where B and H stand for the Brillouin-flow and hollow beam, respectively. Their impedance parameters, identified similarly, are related as follows:

$$\frac{K_B}{K_H} = \left(\frac{Y_c}{Y_B}\right)_{\beta_{0n}} = \left[\frac{2}{\beta b}\left(\frac{\omega - \beta u}{\omega - \beta u - n\dot{\theta}}\right)^2 \frac{I_{nb}'}{I_{nb}}\right]_{\beta_{0n}}, \tag{52}$$

where β_{0n} is the zero of Y_c, i.e., the empty-helix propagation constant. It is found by putting δ_0 to zero:

$$\frac{I_{na}'K_{na}'}{I_{na}K_{na}} = -\left[\frac{\beta^2a^2 + n\beta a \cot\Psi}{ka\beta a \cot\Psi}\right]^2. \tag{53}$$

This is the determinantal equation of the empty sheath helix given by Sensiper,[9] modified by the slow-wave approximation. Sensiper has shown that, for $\beta a > 2$, and in the nondispersive region of the helix, the cold-helix propagation constant is given to a good approximation by

$$\beta_{0n} \cong \beta_0 + \frac{2\pi n}{p}, \tag{54}$$

where β_0 is the propagation constant of the fundamental and p is the helix pitch. Sensiper has also evaluated the impedance of the sheath helix in the nth order mode, at a radius $b < a$, as follows:

$$K_H(n, b) \cong \frac{30}{|\, n + ka \,|} \frac{{I_{nb}}^2}{{I_{na}}^2} \text{ ohms} \tag{55}$$

when $(k/\beta) < 0.4$ and $|\, n \,| > 0$. This is the same as K_H in (52) for the thin hollow beam in confined flow at radius b.

In expression (52) for K_B/K_H there is a factor dependent on the beam velocity components $\dot{\theta}$ and u, arising from the comparison of a rotating with a nonrotating beam. When both beams have the same value of βu, this factor is greater than unity for positive n, indicating that the angular component of beam motion contributes to field-wave interaction in the rotating beam, i.e., to interaction with E_θ as well as E_z and E_r. The remaining terms express, on the other hand, the superior efficiency of the hollow beam due to its concentration in a region of nonzero field.

It will be shown below that the space-charge reduction factor for the $n = 0$ mode is very nearly the same in the presence of a sheath helix as that of a drift tube at the same radius, when the slow wave is in synchronism with the cold-helix propagation constant. Without proof, it seems reasonable to assume that the same equivalence is true for a high-order mode as well. With this assumption, and using the approximation $p_n \cong 0.707$, the impedance ratio can be simplified further as follows:

$$\beta_s \cong \beta_e - \left(\frac{n-1}{2}\right) \beta_c = \beta_{0n}\,, \tag{56}$$

$$\frac{K_B}{K_H} \cong (n-1)^2 \left[\frac{2}{\beta b} \frac{I_{nb}'}{I_{nb}}\right]_{\beta_{0n}}. \tag{57}$$

This, combined with expression (55) for K_H, yields K_B for the comparable Brillouin beam at synchronism.

The sign of n is positive for a wave which spirals in the same sense as the beam, since both n and $\dot{\theta}$ are referred to the same set of cylindrical coordinates. Thus, β_n is less than β_0 when $n\dot{\theta}$ is positive. For the spatial harmonics of an empty helix, the opposite convention has been established; i.e., $\beta_n < \beta_0$ when n and the pitch p have opposite signs. Aside from this distinction, however, there is a close analogy between the spatial harmonic waves on a helix and those on the Brillouin-flow or hollow rotating beam.

The above expression for K_B/K_H is only valid for $n = 0$, and for $n > 1$ whenever the approximation $p_n \cong 0.707$ is valid. The coupling impedance K_B, therefore, is not necessarily zero when $n = 1$. For negative n, the phase constant β is greater than for zero or positive n, and K_B decreases very rapidly as β increases. (This might be expected, as the larger β is, the more rapidly the field decays radially away from the helix.) Thus, none of the negative-order beam harmonics have appreciable coupling impedances, but those of positive order greater than unity may have very large coupling impedances.

Evidence of interaction between a number of these beam hamonics and those of a bifilar helix has been reported by V. P. Kiryushin.[3, 4] Operating a backward-wave oscillator with its electron gun in a field-free region, he found narrow-band gaps in the output spectrum of the tube (and corresponding peaks in the starting current) at a number of discrete values of ω_c/ω, which he attributed to loss of energy to various harmonic modes. The values of ω_c/ω at which these disturbances were noted were found by Kiryushin to correspond to the "ratios of small integers", and appear to show interaction between beam and helix modes of the same as well as of different azimuthal periodicities (when their axial phase velocities are the same). In the latter case, it seems likely that the coupling impedance K_B would lack the factor $(n - 1)^2$ expressing interaction with the azimuthal electric field.

IV. PLASMA FREQUENCY REDUCTION FACTOR FOR BEAM IN SHEATH HELIX

For axial-symmetric waves on a solid-cylindrical beam in confined flow, Branch[10] has found the space-charge reduction factor to be nearly the same in a drift tube as in a helix of the same diameter. A similar computation can be made for the Brillouin-flow beam.

For any beam in a concentric helix, the relation derived by Branch is

$$p^2 = (QK) \frac{(\beta b)^2 V_0^{1/2}}{(174.1)^2}, \tag{58}$$

which reduces, when the beam is at synchronous velocity, to

$$p^2 = (QK) \frac{k}{60\beta_0}. \tag{59}$$

Here Q and K are Pierce's[7] normal-mode parameters, properly evaluated for the finite-diameter beam in question, and V_0 is the dc beam potential. The Q and K values for the Brillouin beam will be identified as before by the subscript B, and those for the thin hollow beam at the bounding radius b by the subscript H.

As shown in the preceding section of this paper,

$$Q_B = Q_H \tag{60}$$

for space-charge modes of any order number, including zero. Fletcher[8] has shown, in curves reproduced in Fig. A6.1 of Ref. 7, that Q_H differs very little from its value in a drift tube of the same diameter ($2a$) as the helix

$$Q_H = \frac{60}{F^3(\beta a)}\left[\frac{K_{0b}}{I_{0b}} - \frac{K_{0a}}{I_{0a}}\right], \tag{61}$$

where $F^3(\beta a)$ is given by Equation (41), p. 232, of Ref. 7. The impedance parameter K_B of the Brillouin-flow beam is related to that of the thin beam at the axis, K_T, as follows:[1]

$$\left(\frac{K_B}{K_T}\right)_{n=0} = \left[\frac{2I_{1b}I_{0b}}{\beta b}\right]_{\beta_0}, \tag{62}$$

where

$$K_T \cong \frac{\beta}{2k} F^3(\beta a). \tag{63}$$

Thus, when the beam is at synchronous velocity,

$$p^2 = \beta b I_{1b} I_{0b}\left[\frac{K_{0b}}{I_{0b}} - \frac{K_{0a}}{I_{0a}}\right], \tag{64}$$

an expression identical with that for p^2 when the beam is in a drift tube of diameter $2a$.[1]

Paschke[11] has questioned the results of Branch's computations for the solid-cylindrical confined-flow beam, on the grounds that QK was computed in terms of an equivalent hollow beam—an equivalence of rather restricted validity. This objection does not apply to the present computation. Since its ac convection current is almost entirely carried by the moving surface charge, the Brillouin-flow beam very closely resembles the thin hollow beam on which the calculation is based.

V. INTERCEPTION NOISE DUE TO IMMERSED GRID

Ashkin and White[5] have obtained a series of periodic noise patterns along a drifting cylindrical beam, by means of an axial-symmetric cavity trailing in the wake of a moving, immersed grid. In addition, they were able to observe changes in beam structure with the aid of Ashkin's beam analyzer,[12] mounted behind the cavity. The beam was produced

by a convergent, shielded gun, and focused by a uniform axial magnetic field in the drift region. The cathode flux could be varied by an auxiliary coil near the cathode.

Some of these observations can be explained on the basis of (a) a general description of the nature of interception noise at microwave frequencies, given by Beam[13] and (b) the nature of higher-order space-charge waves in beams produced by partly-shielded guns, as described in the first section of this paper. This explanation will apply to the periodic noise patterns obtained with the pickup cavity located half a plasma wavelength (in the fundamental mode) behind the moving grid, which fall roughly into two groups:

i. When the fields were adjusted to obtain clear images of the cathode region on the analyzer (22 to 36 gauss at the cathode), the beam was rippled and the noise-current pattern had sharp dips. Within the accuracy of measurement, these dips appeared to coincide with the image planes, and were spaced a cyclotron wavelength apart.

ii. When the fields were adjusted for maximum beam transmission through the gun anode (well below 10 gauss at the cathode), no cathode images were observed and the noise current varied sinusoidally, with large amplitude and the cyclotron period. The beam was comparatively smooth; i.e., its ripple was insufficient to account for the observed noise variations by variations in coupling to the cavity or in intercepted current.

5.1 *Sources of Interception Noise*

When a filamentary electron stream in a finite magnetic field is partially intercepted by a grid, Beam[13] has shown that the transmitted filament contains four uncorrelated noise components: the incident noise current reduced by the transmission factor, plus the incident axial-velocity fluctuations, and, in addition, two new independent fluctuation sources, partition velocity and current, which are due to the uncertainty of electron position at the grid plane or the randomness of interception. The first two components of interception noise, therefore, are produced by the noise space-charge *waves* in the incident beam, whereas the latter two components are due to the behavior of the *particles* in that beam. The latter components arise because of transverse thermal velocities which are uncorrelated with the longitudinal ones; they would be absent in confined flow.

The beam of *finite area* is equivalent to a bundle of many filamentary streams, whose space-charge waves are coupled to one another. Thus, all of the propagating space-charge modes are involved in transporting the current and velocity fluctuations. (The question of the completeness of these modes, in the mathematical sense, does not enter here;[14] it is only necessary that all of the propagating modes be known.) Since the transverse distribution of each incident mode is distinct and unlike the nearly uniform distribution of partition components due to random interception at a grid, each incident mode contributes differently to each of the transmitted modes.

An additional complication that usually besets beams in finite magnetic fields, as Robinson and Kompfner[15] have shown, is an increased spread in longitudinal velocities over the beam area, and increased transverse electron excursions, due to electron-optical defects in beam focusing. For a strongly rippled beam which is alternately focused and defocused Herrmann[16] has shown how the transverse thermal excursions wax and wane along the beam. Increased transverse excursions correspond to increased current partition noise, whereas increased spread in longitudinal velocities means increased velocity partition noise.

Despite the complexity of this description, some general conclusions may be drawn relevant to the Ashkin-White observations:

i. Due to nonlinear mode conversion at an immersed grid, the noise current in the fundamental transmitted mode will depend on all of the propagating modes in the incident beam.

ii. The amplitude of noise current induced in the axial-symmetric cavity, a half-plasma-wavelength behind the grid, will depend chiefly on two factors in the incident beam: the current amplitudes of all the space-charge modes and the transverse excursions of electrons in the incident beam.

5.2 *Noise Modes in Imperfect Brillouin-Flow Beam*

Electron beams ordinarily obtained in the laboratory with incompletely shielded, convergent guns are known to depart considerably from the models assumed in space-charge-wave computations. Thermal electron motions, gas ions and haphazard focusing usually conspire to produce a rippled beam with more or less nonlaminar flow.[16,17] Nevertheless, there is experimental evidence that space-charge waves in such beams closely resemble those predicted for the idealized model with the same average velocity field.

The writer, for example, has found that, in just such an imperfect beam, the radial distribution of ac current density corresponded closely to that calculated for Brillouin flow.[18] In addition, the measured space-charge wavelength (for the fundamental mode) was found to agree closely with the calculated values, based on the average diameter of the usually rippled beam, for field strengths ranging from below the nominal "Brillouin field" to several times that value. Good agreement between measurements and these calculations has also been reported by Winslow[19] for a gap-excited 10-kilovolt beam of microperveance one.

The reason is that the space-charge waves are not dependent on the individual electron trajectories (which may intercept the axis regularly or not[16, 17] in a rippled beam) but only on the net motion of the charge assemblage. Over a considerable range of field strengths, the average beam diameter varies inversely with the field, so that the average plasma frequency remains proportional to the cyclotron frequency over that range, just as in ideal Brillouin flow. When magnetic flux threads the cathode, the field distribution at any cross-section plane of a rippled beam will depart from that in a smooth beam due to (a) the ripple itself, and (b) nonlaminar flow. The former condition causes the angular velocity $\dot{\theta}$ of the smoothed-out charge to vary from plane to plane *along* the beam, whereas the latter causes $\dot{\theta}$ to vary with radius *inside* of the beam. The ensuing field distortion in both cases is periodic along the rippled beam, however, and for relatively small cathode flux or ripple is not likely to produce marked changes in the space-charge wavelength (relative to that in a comparable smooth beam in laminar flow, with the same cathode flux and average beam diameter).

In another set of relevant observations, Ashkin[20] has excited such a beam in the $n = 1$ and $n = 2$ modes, respectively, by means of cavities with the appropriate angular periodicities, and then traced the spiral loci of the current minima along the beam by means of similar pick-up cavities. In each case, the current minima were found to follow the computed axial and rotational fluid, or average, velocities of the beam, in agreement with the description of such waves in the first section of this paper.

When the cathode of such a beam is shielded, the field pattern for any mode is essentially the same in the diode and drift regions. (For small values of the flux parameter α, the transverse field distribution is only slightly different from that in Brillouin flow, and the space-charge wavelength is slightly smaller.) As nearly all such mode-pairs but the fundamental have the same standing-wave periodicity (in both diode and drift regions), and are initially excited at the same plane near the cathode,

they will preserve phase coherence along the axis even after partial energy interchange among the modes in the diode. In the Ashkin-White experiment, therefore, *nearly all of the high-order modes in the beam striking the grid have the same current-minimum planes, spaced a cyclotron wavelength or so apart.*

As primary current fluctuations can occur in an infinitesimally small area, the modal distribution of noise currents is probably nearly flat The sum of the squared moduli of all but the fundamental mode should then greatly outweigh that of the latter alone; and the same should be true of their net contribution to noise current in the fundamental mode, excited at the grid. In a relatively smooth beam, therefore, in which the electron-interception probability is independent of grid position, the cavity-detected noise current should vary sinusoidally and with the cyclotron period, at any frequency. This was the pattern observed under such conditions by Ashkin and White at both 400 and 4000 mc.

When the fields were adjusted for sharp cathode images, the flux parameter α ranged from $\frac{1}{4}$ to $\frac{1}{2}$, and the beam was strongly rippled. Both factors helped to minimize the transverse thermal excursions at the image planes, and thereby the contribution of random interception to partition noise there. If these planes are, in addition, made to coincide with those of noise-current minima for the high-order modes, the observed noise dips should be very much sharper than in the smooth beam, again as observed. As the variation along the beam of partition noise due to random interception is very large in rippled beams with periodic imaging of the cathode, the sharp noise dips are primarily due to such variations, rather than to current variations in the noise standing waves.

The two groups of noise patterns, therefore, illustrate the dual nature of the sources of interception noise at the grid. In the smooth beam, the variations are chiefly due to noise current variations in the space-charge *waves*, whereas in the rippled beam with periodic cathode images they are chiefly due to uncorrelated transverse *particle* excursions of thermal origin.[16] Both processes sometimes happen to have the same axial periodicity, but they are otherwise distinct and independent of one another.

VI. CONCLUSIONS

The higher-order modes of slow space-charge waves on beams produced by shielded or partly shielded cathodes have azimuthal, but no

radial, periodicity. Another feature that distinguishes these waves from those in cylindrical confined-flow beams is that their axial phase velocities increase rapidly with the order number, n, and angular velocity, $\dot{\theta}$, of the rotating beam. With suitable means for exciting such modes, therefore, it should be possible to achieve interaction between a relatively low-voltage beam and the field of a structure with high phase velocity. This should be equally possible for (a) hollow rotating beams, focused in any way whatever, provided they are stable and (b) solid-cylindrical beams produced by shielded guns, with arbitrarily strong focusing fields (since only the net angular velocity of the beam, not the particle trajectories, affects the wave velocity). More generally, the same type of interaction should be possible with stable beams of any geometry, when they have a transverse velocity component parallel to the beam surface and are excited by RF fields which are periodic in that transverse direction.

Another interesting property of harmonic waves on a Brillouin-flow beam is that, because the axial and radial propagation constants are equal, the rate of decay of fields with distance from an enclosing RF structure can be smaller, the smaller this constant is. A computation indicates that, consequently, the coupling of this beam to the harmonic fields of a sheath helix can be quite large. Experimental evidence of such interaction has been reported.[4]

When a Brillouin-flow beam is at synchronous velocity inside a concentric sheath helix, its plasma-frequency reduction factor in the fundamental mode has been found to be the same as if the beam were in a drift tube of the same diameter as the helix.

The computations also show that, for nearly all of the higher-order space-charge modes on beams from shielded or nearly-shielded guns, the space-charge wavelength is close to twice the cyclotron wavelength. This feature, together with a multimode description of interception noise given by Beam,[13] has helped explain some periodic noise patterns obtained with a cavity behind an immersed grid.[5]

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